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ON RANDOM COINCIDENCE POINT AND RANDOM COUPLED FIXED POINT THEOREMS IN ORDERED METRIC SPACES

R. A. Rashwan and H. A. Hammad

Department of Mathematics
Faculty of Science
Assuit University
Assuit 71516, Egypt
e-mail: rr_rashwan54@yahoo.com

Department of Mathematics
Faculty of Science
Sohag University
Sohag 82524, Egypt
e-mail: h_elmagd89@yahoo.com

Abstract

In this paper, we obtain random coupled coincidence point theorems in an ordered metric space X for a pair of random mappings $F : \Omega \times (X \times X) \to X$ and $g : \Omega \times X \to X$ under certain contractive conditions which are commutative under generalized altering distance function in five variables. Random coupled fixed point results (as corollaries) under certain contractive conditions will be excerpted from our theorems. We also support our result by an example.

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1. Introduction

The theory of random mappings is a substantial branch of probabilistic analysis which plays pivotal role in many applied mathematics. Random fixed point theorems for contraction mappings on various spaces have been established by several authors [13, 14, 16-22, 27, 33, 41, 44, 45].

In 1994, the notion of partial metric spaces was initiated by Matthews [28]. He extended the Banach contraction principle from metric spaces to partial metric spaces.

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

One of the recent trends, initiated by Ran and Reurings [37], in fixed point theory, is to study the existence and uniqueness of certain operators in the context of partially ordered metric spaces (see for example, [2, 1, 4, 5, 15, 26, 31, 32]).

In 1984, Khan et al. [24] introduced the notion of an altering distance function for one variable as the following:

Definition 1.2. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called *altering distance function* if and only if

(1) φ is continuous,

(2) φ is nondecreasing,

(3) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Several authors worked in this notion and established nice results for applied it to obtaining fixed point results in metric spaces.

In recent years, Choudhury and Dutta [8] generalized this notion to a two-variable function and it was also extended to three-variable function by Choudhury [9] as the following:

Definition 1.3. A function $\varphi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to be a *generalized altering distance function* if and only if

(1) φ is continuous,

(2) φ is nondecreasing in all the three variables,

(3) $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0.$

In 2008, Rao et al. [36] introduced the altering distance function in five variables to generalize the results of Choudhury [9] as follows:

Definition 1.4. A function

$$\varphi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$$

is said to be a *generalized altering distance function* if and only if

(1) φ is continuous,

(2) φ is nondecreasing in all the five variables,

(3) $\varphi(x, y, z, w, t) = 0 \Leftrightarrow x = y = z = w = t = 0.$

In 2006, the concept of coupled fixed point was presented by Bhaskar and Lakshmikantham [5]. They studied some nice coupled fixed point theorems, after three years, Lakshmikantham and Ćirić [26] introduced the notion of a coupled coincidence point and coupled fixed point of mappings. Many theorems under this title were recognized, see [10-12, 23, 39, 40, 42, 43, 3, 34, 35].

Recall that if $(X, \leq)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be *nondecreasing*. Similarly, a nonincreasing mapping is defined.

The same authors [5] introduced the following notions of a mixed monotone mapping as:

Definition 1.5. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to has the *mixed monotone property* if F is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.6 [5]**.** An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

The concept of the mixed monotone property is generalized in [26] as:

Definition 1.7. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$, $g : X \to X$. The mapping F is said to has the *mixed g-monotone property* if F is monotone g-nondecreasing in x and is monotone g-nonincreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Clearly, if g is the identity mapping, then Definition 1.7 reduces to Definition 1.5.

Definition 1.8 [26]**.** An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x) \text{ and } F(y, x) = g(y).$$

Definition 1.9 [1]**.** Let X be a nonempty set, $F : X \times X \to X$ and $g : X \to X$. Then F and g are commutative if for all $x, y \in X$,

$$g(F(x, y)) = F(g(x), g(y)).$$

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω, (X, d) stands for a complete metric space.

Definition 2.1 [6]**.** A mapping $f : \Omega \to X$ is said to be *measurable* if $f^{-1}(B) \in \Sigma$ for every Borel subset B of X.

Definition 2.2 [7]**.** A mapping $T : \Omega \times X \to X$ is a random operator if for each fixed $t \in X$, the mapping $T(., t) : \Omega \to X$ is measurable.

Definition 2.3 [6]**.** A measurable mapping $\xi : \Omega \to X$ is a random fixed point of a random operator $T : \Omega \times X \to X$ if $T(t, \xi(t)) = \xi(t)$ for each $t \in \Omega$.

Definition 2.4 [25]**.** A measurable mapping $\xi : \Omega \to X$ is a random coincidence of the random operators $T : \Omega \times X \to X$ and $g : \Omega \times X \to X$ if $T(t, \xi(t)) = g(t, \xi(t))$ for each $t \in \Omega$.

Definition 2.5 [25]**.** Let (X, d) be a separable metric space, (Ω, Σ) be a measurable space and $T : \Omega \times (X \times X) \to X$, $g : \Omega \times X \to X$. Then T and g are commutative if

$$T(\omega, (g(\omega, x), g(\omega, y))) = g(\omega, T(w, (x, y)))$$

for all $\omega \in \Omega, x, y \in X$.

3. Main Results

The following lemma is used in the sequel:

Lemma 3.1. *Let (X, d) be a metric space and let $\{g(y_n)\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(g(y_n), g(y_{n-1})) = 0. \tag{1}$$

Suppose that $\{g(y_n)\}$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and two sequences of positive integers $\{n(k)\}$, $\{m(k)\}$, $m(k) > n(k) \geq k$ such that the following four sequences tend to ε:

$$d(g(y_{m_k}), g(y_{n_k})), \quad d(g(y_{m_k}), g(y_{n_k+1})),$$

$$d(g(y_{m_k-1}), g(y_{n_k})), \quad d(g(y_{m_k-1}), g(y_{n_k+1})).$$

Proof. Suppose that $\{g(y_n)\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences of positive integers $\{n(k)\}$, $\{m(k)\}$ for each integer k such that

$$d(g(y_{n_k}), g(y_{m_k})) \geq \varepsilon, \tag{2}$$

let m_k be the least integer exceeding n_k with $m(k) > n(k) \geq k$, satisfying (2). It follows that

$$d(g(y_{n_k}), g(y_{m_k-1})) < \varepsilon, \tag{3}$$

from the triangle inequality, we can write

$$d(g(y_{n_k}), g(y_{m_k})) \leq d(g(y_{n_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k})),$$

$$|d(g(y_{n_k+1}), g(y_{m_k})) - d(g(y_{n_k}), g(y_{m_k}))| \leq d(g(y_{n_k+1}), g(y_{n_k})),$$

$$|d(g(y_{n_k}), g(y_{m_k-1})) - d(g(y_{n_k}), g(y_{m_k}))| \leq d(g(y_{m_k-1}), g(y_{m_k})),$$

$$|d(g(y_{n_k+1}), g(y_{m_k-1})) - d(g(y_{n_k+1}), g(y_{m_k}))| \leq d(g(y_{m_k-1}), g(y_{m_k})). \tag{4}$$

Taking the limit as $k \to \infty$ in terms of (2)-(4) and using (1), we have

$$\varepsilon = \lim_{k\to\infty} d(g(y_{n_k}), g(y_{m_k})) = \lim_{k\to\infty} d(g(y_{n_k+1}), g(y_{m_k}))$$

$$= \lim_{k\to\infty} d(g(y_{n_k+1}), g(y_{m_k-1})) = \lim_{k\to\infty} d(g(y_{n_k}), g(y_{m_k-1})).$$

This completes the proof of Lemma 3.1.

Remark 3.1. If we put $g = I$, where I is the identity mapping in Lemma 3.1, we have Lemma 10 of [30].

The following theorem is a generalized version Theorem 11 of [30].

Theorem 3.1. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space. Suppose* $F: X \times X \to X$ *and* $g: X \to X$ *are such that F has the mixed g-monotone property and*

$$\Phi_1(d(F(x, y), F(u, v)))$$
$$\leq \Psi_1\begin{pmatrix} d(gx, gu), d(gy, gv), d(gx, F(x, y)), d(gu, F(u, v)), \\ \frac{1}{2}[d(gx, F(u, v)) + d(gu, F(x, y))] \end{pmatrix}$$
$$- \Psi_2\begin{pmatrix} d(gx, gu), d(gy, gv), d(gx, F(x, y)), d(gu, F(u, v)), \\ \frac{1}{2}[d(gx, F(u, v)) + d(gu, F(x, y))] \end{pmatrix}, \quad (5)$$

for $x, y, u, v \in X$, *for which* $g(x) \geq g(u)$ *and* $g(y) \leq g(v)$, *where* Ψ_1, Ψ_2 *are generalized altering distance functions and* $\Phi_1(x) = \Psi_1(x, x, x, x, x)$. *Suppose* $F(X \times X) \subseteq g(X)$, *where g is continuous and commutes with F and suppose either*

(a) *F is continuous or*

(b) *X has the following properties*:

(i) *if a nondecreasing sequence* $x_n \to x$, *then* $x_n \leq x$ *for all n*,

(ii) *if a nonincreasing sequence* $y_n \to y$, *then* $y \leq y_n$ *for all n*.

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \leq F(x_0, y_0)$ *and* $g(y_0) \geq F(y_0, x_0)$, *then there exists* $(x, y) \in X$ *such that* $F(x, y) = x$ *and* $F(y, x) = y$.

That is, F and g have a coupled coincidence.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that

$g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again, from $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. In this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \text{ for all } n \geq 0. \tag{6}$$

By mathematical induction, one can prove (see Lakshmikantham [26, pp. 4343-4344]):

$$g(x_n) \leq g(x_{n+1}) \text{ and } g(y_n) \geq g(y_{n+1}) \text{ for all } n \geq 0. \tag{7}$$

From (7) and (6), we have

$$\begin{aligned}
&\Phi_1(d(gx_{n+1}, gx_n)) \\
&= \Phi_1(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\
&\leq \Psi_1\begin{pmatrix} d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, F(x_n, y_n)), \\ d(gx_{n-1}, F(x_{n-1}, y_{n-1})), \frac{1}{2}[d(gx_n, F(x_{n-1}, y_{n-1})) \\ + d(gx_{n-1}, F(x_n, y_n))] \end{pmatrix} \\
&\quad - \Psi_2\begin{pmatrix} d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, F(x_n, y_n)), \\ d(gx_{n-1}, F(x_{n-1}, y_{n-1})), \frac{1}{2}[d(gx_n, F(x_{n-1}, y_{n-1})) \\ + d(gx_{n-1}, F(x_n, y_n))] \end{pmatrix} \\
&= \Psi_1\begin{pmatrix} d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n), \\ \frac{1}{2}d(gx_{n-1}, gx_{n+1}) \end{pmatrix} \\
&\quad - \Psi_2\begin{pmatrix} d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n), \\ \frac{1}{2}d(gx_{n-1}, gx_{n+1}) \end{pmatrix} \\
&\leq \Psi_1\begin{pmatrix} d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n), \\ \frac{1}{2}d(gx_{n-1}, gx_{n+1}) \end{pmatrix},
\end{aligned} \tag{8}$$

since Ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$,

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1}), \tag{9}$$

again from (7) and (6), we get

$$\begin{aligned}
&\Phi_1(d(gy_{n+1}, gy_n)) \\
&= \Phi_1(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\
&\leq \Psi_1\begin{pmatrix} d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), d(gy_n, F(y_n, x_n)), \\ d(gy_{n-1}, F(y_{n-1}, x_{n-1})), \frac{1}{2}[d(gy_n, F(y_{n-1}, y_{n-1})) \\ + d(gy_{n-1}, F(y_n, x_n))] \end{pmatrix} \\
&\quad - \Psi_2\begin{pmatrix} d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), d(gy_n, F(y_n, x_n)), \\ d(gy_{n-1}, F(y_{n-1}, x_{n-1})), \frac{1}{2}[d(gy_n, F(y_{n-1}, y_{n-1})) \\ + d(gy_{n-1}, F(y_n, x_n))] \end{pmatrix} \\
&= \Psi_1\begin{pmatrix} d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n), \\ \frac{1}{2}d(gy_{n-1}, gy_{n+1}) \end{pmatrix} \\
&\quad - \Psi_2\begin{pmatrix} d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n), \\ \frac{1}{2}d(gy_{n-1}, gy_{n+1}) \end{pmatrix} \\
&\leq \Psi_1\begin{pmatrix} d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n), \\ \frac{1}{2}d(gy_{n-1}, gy_{n+1}) \end{pmatrix},
\end{aligned} \tag{10}$$

since Ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$,

$$d(gy_{n+1}, gy_n) \leq d(gy_n, gy_{n-1}). \tag{11}$$

In view of (9) and (11), the sequences $\{d(gx_{n+1}, gx_n)\}$ and $\{d(gy_{n+1}, gy_n)\}$ are nonincreasing, so there exist $\alpha \geq 0$ and $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = \alpha \text{ and } \lim_{n\to\infty} d(gy_{n+1}, gy_n) = \gamma.$$

Again, since Ψ_1 is monotone increasing with respect to the fifth variable, from (8), we have by triangular inequality

$$d(gx_{n+1}, gx_n) \leq \frac{1}{2} d(gx_{n-1}, gx_{n+1}) \leq \frac{1}{2} d(gx_{n-1}, gx_n) + \frac{1}{2} d(gx_n, gx_{n+1}),$$

by taking the limit, we have $\lim_{n\to\infty} d(gx_{n-1}, gx_{n+1}) = 2\alpha,$ similarly from (10), $\lim_{n\to\infty} d(gy_{n-1}, gy_{n+1}) = 2\gamma.$

Also, taking the limit as $n \to \infty$ in (8) and (10), respectively, and using the continuity of Ψ_1 and Ψ_2, we get

$$\Phi_1(\alpha) \leq \Psi_1(\alpha, \gamma, \alpha, \alpha, \alpha) - \Psi_2(\alpha, \gamma, \alpha, \alpha, \alpha) \tag{12}$$

and

$$\Phi_1(\gamma) \leq \Psi_1(\gamma, \alpha, \gamma, \gamma, \gamma) - \Psi_2(\gamma, \alpha, \gamma, \gamma, \gamma). \tag{13}$$

Assume that $\alpha \neq \gamma.$ Without loss of generality, suppose that $\gamma < \alpha$ and using $\Phi_1(x) = \Psi_1(x, x, x, x, x),$ so

$$\begin{aligned} \Phi_1(\alpha) &\leq \Psi_1(\alpha, \gamma, \alpha, \alpha, \alpha) - \Psi_2(\alpha, \gamma, \alpha, \alpha, \alpha) \\ &\leq \Phi_1(\alpha) - \Psi_2(\alpha, \gamma, \alpha, \alpha, \alpha), \end{aligned} \tag{14}$$

which holds unless $\Psi_2(\alpha, \gamma, \alpha, \alpha, \alpha) = 0,$ that is, $\alpha = \gamma,$ a contradiction.

We deduce that

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0 \text{ and } \lim_{n\to\infty} d(gy_n, gy_{n-1}) = 0. \tag{15}$$

Now we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{g(x_n)\}$ or $\{g(y_n)\}$ is not a Cauchy sequence. Then, by virtue of Lemma 3.1, the sequences

$$d(gx_{m_k}, gx_{n_k}), d(gx_{m_k}, gx_{n_k+1}), d(gx_{m_k-1}, gx_{n_k}), d(gx_{m_k-1}, gx_{n_k+1}) \tag{16}$$

and

$$d(gy_{m_k}, gy_{n_k}), d(gy_{m_k}, gy_{n_k+1}), d(gy_{m_k-1}, gy_{n_k}), d(gy_{m_k-1}, gy_{n_k+1}) \quad (17)$$

tend to ε when $k \to \infty$. It follows that

$$\lim_{k\to\infty} d(gx_{n_k-1}, gx_{m_k+1}) \le \varepsilon \text{ and } \lim_{k\to\infty} d(gy_{n_k-1}, gy_{m_k+1}) \le \varepsilon. \quad (18)$$

Since $m(k) \ge n(k) - 1$, so from (7) and (6), we have

$$\Phi_1(d(gx_{m(k)+1}, gx_{n(k)}))$$

$$= \Phi_1(d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)-1}, y_{n(k)-1})))$$

$$\le \Psi_1\begin{pmatrix} d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\ d(gx_{m(k)}, F(x_{m(k)}, y_{m(k)})), d(gx_{n(k)-1}, F(x_{n(k)-1}, y_{n(k)-1})), \\ \frac{1}{2}[d(gx_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1})) + d(gx_{n(k)-1}, F(x_{m(k)}, y_{m(k)}))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\ d(gx_{m(k)}, F(x_{m(k)}, y_{m(k)})), d(gx_{n(k)-1}, F(x_{n(k)-1}, y_{n(k)-1})), \\ \frac{1}{2}[d(gx_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1})) + d(gx_{n(k)-1}, F(x_{m(k)}, y_{m(k)}))] \end{pmatrix}. \quad (19)$$

In addition, we have

$$\Phi_1(d(gy_{n(k)}, gy_{m(k)+1}))$$

$$= \Phi_1(d(F(y_{n(k)-1}, x_{n(k)-1}), F(y_{m(k)}, x_{m(k)})))$$

$$\le \Psi_1\begin{pmatrix} d(gy_{n(k)-1}, gy_{m(k)}), d(gx_{n(k)-1}, gx_{m(k)}), \\ d(gy_{m(k)}, F(gy_{m(k)}, gx_{m(k)})), d(gy_{n(k)-1}, F(y_{n(k)-1}, x_{n(k)-1})), \\ \frac{1}{2}[d(gy_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1})) + d(gy_{n(k)-1}, F(y_{m(k)}, x_{m(k)}))] \end{pmatrix}$$

$$- \Psi_2 \begin{pmatrix} d(gy_{n(k)-1}, gy_{m(k)}), d(gx_{n(k)-1}, gx_{m(k)}), \\ d(gy_{m(k)}, F(y_{m(k)}, x_{m(k)})), d(gy_{n(k)-1}, F(y_{n(k)-1}, x_{n(k)-1})), \\ \frac{1}{2}[d(gy_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1})) + d(gy_{n(k)-1}, F(y_{m(k)}, x_{m(k)}))] \end{pmatrix}. \tag{20}$$

Taking the limit as $k \to \infty$ using (16) to (18) and the continuity of Ψ_1 and Ψ_2 in (19), we obtain

$$\Phi_1(\varepsilon) \leq \Psi_1(\varepsilon, \varepsilon, 0, 0, \varepsilon) - \Psi_2(\varepsilon, \varepsilon, 0, 0, \varepsilon) \leq \Phi_1(\varepsilon) - \Psi_2(\varepsilon, \varepsilon, 0, 0, \varepsilon), \tag{21}$$

this holds if $\Psi_2(\varepsilon, \varepsilon, 0, 0, \varepsilon) = 0,$ this implies that $\varepsilon = 0,$ a contradiction, since $\varepsilon > 0.$ We deduce that $\{g(x_n)\}$ is a Cauchy sequence. Similarly, taking the limit as $k \to \infty$ in (20), we also get $\{g(y_n)\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, so there exist points x and y in X such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y. \tag{22}$$

From (22) and continuity of g,

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \text{ and } \lim_{n\to\infty} g(g(y_n)) = g(y). \tag{23}$$

From (6) and commutativity of F and g,

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)) \text{ and}$$

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{24}$$

We now show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Consider the assumption (a) holds, taking the limit as $n \to \infty$ in (24), by (22), (23) and continuity of F, we have

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} F(g(x_n), g(y_n))$$

$$= F(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)) = F(x, y),$$

$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} F(g(y_n), g(x_n))$$

$$= F(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(x_n)) = F(y, x).$$

Thus, we proved that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Suppose now that (b) holds, from (7), $g(x_n)$ is nondecreasing and $g(x_n) \to x$ and $g(y_n)$ is nonincreasing and $g(y_n) \to y$, from (i) and (ii), we have $g(x_n) \leq x$ and $g(y_n) \geq y$ for all *n*. Then, by triangle inequality and (24), we have

$$d(F(x, y), g(x)) \leq d(F(x, y), g(g(x_{n+1}))) + d(g(g(x_{n+1})), g(x))$$

$$\leq d(F(x, y), F(g(x_n), g(y_n))) + d(g(g(x_{n+1})), g(x)).$$

Then, from (5) we have

$$\Phi_1(d(F(x, y), g(x)))$$

$$= \Phi_1(d(F(x, y), F(g(x_n), g(y_n)))) + \Phi_1(d(g(g(x_{n+1})), g(x)))$$

$$\leq \Psi_1\begin{pmatrix} d(g(x), g(g(x_n)), d(g(y), g(g(y_n))), d(g(x)F(x, y)), \\ d(g(g(x_n)), F(g(x_n), g(y_n))), \\ \frac{1}{2}[d(g(x), F(g(x_n), g(y_n))) + d(g(x_n), F(x, y))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(x), g(g(x_n)), d(g(y), g(g(y_n))), d(g(x)F(x, y)), \\ d(g(g(x_n)), F(g(x_n), g(y_n))), \\ \frac{1}{2}[d(g(x), F(g(x_n), g(y_n))) + d(g(x_n), F(x, y))] \end{pmatrix}$$

$$+ \Phi_1(d(g(g(x_{n+1})), g(x))),$$

taking the limit in above inequality and using the continuity of Ψ_1 and Ψ_2, we get

$$\Phi_1(d(F(x, y), g(x)))$$

$$\leq \Psi_1(0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), d(g(x), F(x, y)))$$

$$- \Psi_2(0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), d(g(x), F(x, y))) + \Phi_1(0),$$

since $d(F(x, y), g(x)) \geq 0$ and using $\Phi_1(x) = \Psi_1(x, x, x, x, x)$, we get

$$\Phi_1(d(F(x, y), g(x)))$$

$$\leq \Psi_1(0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), d(g(x), F(x, y)))$$

$$- \Psi_2(0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), d(g(x), F(x, y)))$$

$$\leq \Phi_1(d(F(x, y), g(x))) - \Psi_2\begin{pmatrix} 0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), \\ d(g(x), F(x, y)) \end{pmatrix},$$

this holds if

$$\Psi_2(0, 0, d(g(x), F(x, y)), d(g(x), F(x, y)), d(g(x), F(x, y))) = 0,$$

therefore $d(g(x), F(x, y)) = 0$, i.e., $F(x, y) = g(x)$. Similarly, by the same manner, we may show that $F(y, x) = g(y)$.

Thus, (x, y) is a coupled coincidence point of F and g.

The following lemma is useful to shorten the proof of the following theorem.

Lemma 3.2. *Let* (X, d) *be a metric space and let* $\{g(\omega, \eta_n(\omega))\}$ *be a sequence such that*

$$\lim_{n \to \infty} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))) = 0.$$

Suppose that $\{g(\omega, \eta_n(\omega))\}$ *is not a Cauchy sequence. Then there exist an* $\varepsilon > 0$ *and two sequences of positive integers* $\{n(k)\}$, $\{m(k)\}$, $m(k) > n(k) \geq k$ *such that the following four sequences tend to* ε:

$$d(g(\omega, \eta_{m_k}(\omega)), g(\omega, \eta_{n_k}(\omega))), d(g(\omega, \eta_{m_k}(\omega)), g(\omega, \eta_{n_k+1}(\omega))),$$

$$d(g(\omega, \eta_{m_k-1}(\omega)), g(\omega, \eta_{n_k}(\omega))), d(g(\omega, \eta_{m_k-1}(\omega)), g(\omega, \eta_{n_k+1}(\omega))).$$

Proof. We can obtain the proof immediately by the same manner of Lemma 3.1.

The following theorem gives random version of Theorem 3.1 for a pair of random mappings $F : \Omega \times (X \times X) \to X$ and $g : \Omega \times X \to X$ under the set of conditions.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose $F : \Omega \times (X \times X) \to X$ and $g : \Omega \times X \to X$ are mappings such that*

(i) $F(\omega, .), g(\omega, .)$ *are continuous for all* $\omega \in \Omega$,

(ii) $F(., v), g(., x)$ *are measurable for all* $v \in X \times X$ *and* $x \in X$,

(iii) $F : \Omega \times (X \times X) \to X$ *and* $g : \Omega \times X \to X$ *are such that F has the mixed g-monotone property and*

$$\Phi_1(d(F(\omega, (x, y)), F(\omega, (u, v))))$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, x), g(\omega, u)), d(g(\omega, y), g(\omega, v)), d(g(\omega, x), g(\omega, (x, y))), \\ d(g(\omega, u), F(\omega, (u, v))), \\ \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, x), g(\omega, u)), d(g(\omega, y), g(\omega, v)), \\ d(g(\omega, x), F(\omega, (x, y))), d(g(\omega, u), F(\omega, (u, v))), \\ \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}, \qquad (25)$$

for $x, y, u, v \in X$, *for which* $g(\omega, x) \geq g(\omega, u)$ *and* $g(\omega, y) \leq g(\omega, v)$, *where* Ψ_1, Ψ_2 *are generalized altering distance functions and* $\Phi_1(x) =$

$\Psi_1(x, x, x, x, x)$. *Suppose* $F(X \times X) \subseteq g(X)$, $g(\omega, X) \to X$ *is continuous and commutes with F*; *and suppose either*

(a) *F is continuous or*

(b) *X has the following properties*:

(I) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n*,

(II) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all n*.

If there exist measurable mappings $\eta_0, \xi_0 : \Omega \to X$ *such that*

$$g(\omega, \xi_0(\omega)) \leq F(\omega, (\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$g(\omega, \eta_0(\omega)) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega))) \text{ for all } \omega \in \Omega,$$

then there are measurable mappings $\varphi, \theta : \Omega \to X$ *such that*

$$d(\omega, \varphi(\omega)) = F(\omega, (\varphi(\omega), \theta(\omega))) \text{ and}$$

$$g(\omega, \theta(\omega)) = F(\omega, (\theta(\omega), \varphi(\omega))) \text{ for all } \omega \in \Omega,$$

that is, *F and g have a random coupled coincidence*.

Proof. Let $\Theta = \{\xi : \Omega \to X\}$ be a family of measurable mappings. Define a function $h : \Omega \times X \to R^+$ as follows:

$$h(\omega, x) = d(x, g(\omega, x)).$$

Since $x \to g(\omega, x)$ is continuous for all $\omega \in \Omega$, we conclude that $h(\omega, .)$ is continuous for all $\omega \in \Omega$. Again, since $\omega \to g(\omega, x)$ is measurable for all $x \in X$, we conclude that $h(., x)$ is measurable for all $\omega \in \Omega$ (see Wagner [45, p. 868]). Thus, $h(\omega, x)$ is a Carathéodory function. Therefore, if $\xi : \Omega \to X$ is a measurable mapping, then $\omega \to h(\omega, \xi(\omega))$ is also measurable (see [44]). Also, for each $\xi \in \Theta$, the function $\eta : \Omega \to X$ defined by $\eta(\omega) = g(\omega, \xi(\omega))$ is measurable, that is, $\eta \in \Theta$. Now we are going to construct two sequences of measurable mappings $\{\eta_n\}$ and $\{\xi_n\}$ in

Θ and two sequences $\{g(\omega, \xi_n(\omega))\}$ and $\{g(\omega, \eta_n(\omega))\}$ in X as follows. Let $\xi_0, \eta_0 \in \Theta$ be such that

$$g(\omega, \xi_0(\omega)) \leq F(\omega, (\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$g(\omega, \eta_0(\omega)) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega)))$$

for all $\omega \in \Omega$. Since $F(\omega, (\xi_0(\omega), \eta_0(\omega))) \in X = g(\omega, X)$, by a short of Filippov measurable implicit function theorem (see [4, 19, 29]), there is $\xi_1 \in \Theta$ such that $g(\omega, \xi_1(\omega)) = F(\omega, (\xi_0(\omega), \eta_0(\omega)))$. Similarly, as $F(\omega, (\eta_0(\omega), \xi_0(\omega))) \in g(\omega, X)$, there is $\eta_1 \in \Theta$ such that $g(\omega, \eta_1(\omega)) = F(\omega, (\eta_0(\omega), \xi_0(\omega)))$. Thus, $F(\omega, (\xi_0(\omega), \eta_0(\omega))), F(\omega, (\eta_0(\omega), \xi_0(\omega)))$, are well defined now. Again, since

$$F(\omega, (\xi_1(\omega), \eta_1(\omega))), F(\omega, (\eta_1(\omega), \xi_1(\omega))) \in g(\omega, X),$$

there are $\xi_2, \eta_2 \in \Theta$ such that

$$g(\omega, \xi_2(\omega)) = F(\omega, (\xi_1(\omega), \eta_1(\omega))) \text{ and}$$

$$g(\omega, \eta_2(\omega)) = F(\omega, (\eta_1(\omega), \xi_1(\omega))).$$

Continuing this process, we can construct sequences $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ in X such that

$$g(\omega, \xi_{n+1}(\omega)) = F(\omega, (\xi_n(\omega), \eta_n(\omega))) \text{ and}$$

$$g(\omega, \eta_{n+1}(\omega)) = F(\omega, (\eta_n(\omega), \xi_n(\omega))). \tag{26}$$

By mathematical induction, we can prove that (see [25, p. 1251])

$$g(\omega, \xi_n(\omega)) \leq g(\omega, \xi_{n+1}(\omega)) \text{ and } g(\omega, \eta_n(\omega)) \geq g(\omega, \eta_{n+1}(\omega)). \tag{27}$$

From (25) and (27), we have

$$\Phi_1(d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega))))$$

$$= \Phi_1(d(F(\omega, (\xi_n(\omega), \eta_n(\omega))), F(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega)))))$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), \\ d(g(\omega, \xi_n(\omega)), F(\omega, (\xi_n(\omega), \eta_n(\omega)))), \\ d(g(\omega, \xi_{n-1}(\omega)), F(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \xi_n(\omega)), F(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega)))) \\ + d(g(\omega, \xi_{n-1}(\omega)), F(\omega, (\xi_n(\omega), \eta_n(\omega))))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), \\ d(g(\omega, \xi_n(\omega)), F(\omega, (\xi_n(\omega), \eta_n(\omega)))), \\ d(g(\omega, \xi_{n-1}(\omega)), F(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \xi_n(\omega)), F(\omega, (\xi_{n-1}(\omega), \eta_{n-1}(\omega)))) \\ + d(g(\omega, \xi_{n-1}(\omega)), F(\omega, (\xi_n(\omega), \eta_n(\omega))))] \end{pmatrix}$$

$$= \Psi_1\begin{pmatrix} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), \\ d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))), d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))), \\ \frac{1}{2}d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_{n+1}(\omega))) \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), \\ d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))), d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))), \\ \frac{1}{2}d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_{n+1}(\omega))) \end{pmatrix}$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), \\ d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))), d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))), \\ \frac{1}{2}d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_{n+1}(\omega))) \end{pmatrix}, \quad (28)$$

since Ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$,

$$d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega))) \leq d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))). \quad (29)$$

Again from (25) and (27), we have

$$\Phi_1(d(g(\omega, \eta_{n+1}(\omega)), g(\omega, \eta_n(\omega))))$$

$$= \Phi_1(d(F(\omega, (\eta_n(\omega), \xi_n(\omega))), F(\omega, (\eta_{n-1}(\omega), \xi_{n-1}(\omega)))))$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), \\ d(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \xi_n(\omega)))), \\ d(g(\omega, \eta_{n-1}(\omega)), F(\omega, (\eta_{n-1}(\omega), \xi_{n-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \eta_n(\omega)), F(\omega, (\eta_{n-1}(\omega), \xi_{n-1}(\omega)))) \\ \quad + d(g(\omega, \eta_{n-1}(\omega)), F(\omega, (\eta_n(\omega), \xi_n(\omega))))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), \\ d(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \xi_n(\omega)))), \\ d(g(\omega, \eta_{n-1}(\omega)), F(\omega, (\eta_{n-1}(\omega), \xi_{n-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \eta_n(\omega)), F(\omega, (\eta_{n-1}(\omega), \xi_{n-1}(\omega)))) \\ \quad + d(g(\omega, \eta_{n-1}(\omega)), F(\omega, (\eta_n(\omega), \xi_n(\omega))))] \end{pmatrix}$$

$$= \Psi_1\begin{pmatrix} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), \\ d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))), d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))), \\ \frac{1}{2}d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_{n+1}(\omega))) \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), \\ d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))), d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))), \\ \frac{1}{2}d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_{n+1}(\omega))) \end{pmatrix}$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))), d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n-1}(\omega))), \\ d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))), d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_n(\omega))), \\ \frac{1}{2}d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_{n+1}(\omega))) \end{pmatrix}, \quad (30)$$

also since Ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$,

$$d(g(\omega, \eta_{n+1}(\omega)), g(\omega, \eta_n(\omega))) \leq d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n-1}(\omega))). \quad (31)$$

In view of (29) and (31), the sequences $\{d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega)))\}$ and $\{d(g(\omega, \eta_{n+1}(\omega)), g(\omega, \eta_n(\omega)))\}$ are nonincreasing, so there exist $\alpha \geq 0$ and $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega))) = \alpha$$

and

$$\lim_{n\to\infty} d(g(\omega, \eta_{n+1}(\omega)), g(\omega, \eta_n(\omega))) = \gamma.$$

Again, since Ψ_1 is monotone increasing with respect to the fifth variable, from (28), we have by triangular inequality

$$d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega)))$$

$$\leq \frac{1}{2} d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_{n+1}(\omega)))$$

$$\leq \frac{1}{2} d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_n(\omega))) + \frac{1}{2} d(g(\omega, \xi_n(\omega)), g(\omega, \xi_{n+1}(\omega))),$$

so in the limit, we have

$$\lim_{n\to\infty} d(g(\omega, \xi_{n-1}(\omega)), g(\omega, \xi_{n+1}(\omega))) = 2\alpha,$$

similarly, from (30),

$$\lim_{n\to\infty} d(g(\omega, \eta_{n-1}(\omega)), g(\omega, \eta_{n+1}(\omega))) = 2\gamma.$$

Passing on the limit $n \to \infty$ in (28) and (30), respectively and using the continuity of Ψ_1 and Ψ_2, we get the same equations (12) and (13). Assume that $\alpha \neq \gamma$. Without loss of generality, suppose that $\gamma < \alpha$ and using

$\Phi_1(x) = \Psi_1(x, x, x, x, x)$, so we get equation (14) which holds unless $\Psi_2(\alpha, \gamma, \alpha, \alpha, \alpha) = 0$, that is, $\alpha = \gamma$, a contradiction.

We deduce that

$$\lim_{n\to\infty} d(g(\omega, \xi_{n+1}(\omega)), g(\omega, \xi_n(\omega))) = 0 \text{ and}$$

$$\lim_{n\to\infty} d(g(\omega, \eta_{n+1}(\omega)), g(\omega, \eta_n(\omega))) = 0. \tag{32}$$

Now, we prove that $\{g(\omega, \xi_n(\omega))\}$ and $\{g(\omega, \eta_{n+1}(\omega))\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{g(\omega, \xi_n(\omega))\}$ or $\{g(\omega, \eta_{n+1}(\omega))\}$ is not a Cauchy sequence. Then, by virtue of Lemma 3.2, the sequences

$$d(g(\omega, \xi_{m_k}(\omega)), g(\omega, \xi_{n_k}(\omega))), d(g(\omega, \xi_{m_k}(\omega)), g(\omega, \xi_{n_k+1}(\omega))),$$

$$d(g(\omega, \xi_{m_k-1}(\omega)), g(\omega, \xi_{n_k}(\omega))), d(g(\omega, \xi_{m_k-1}(\omega)), g(\omega, \xi_{n_k+1}(\omega))) \tag{33}$$

and

$$d(g(\omega, \eta_{m_k}(\omega)), g(\omega, \eta_{n_k}(\omega))), d(g(\omega, \eta_{m_k}(\omega)), g(\omega, \eta_{n_k+1}(\omega))),$$

$$d(g(\omega, \eta_{m_k-1}(\omega)), g(\omega, \eta_{n_k}(\omega))), d(g(\omega, \eta_{m_k-1}(\omega)), g(\omega, \eta_{n_k+1}(\omega))) \tag{34}$$

tend to ε when $k \to \infty$, it follows that

$$\lim_{k\to\infty} d(g(\omega, \xi_{m_k-1}(\omega)), g(\omega, \xi_{n_k}(\omega))) \le \varepsilon \text{ and}$$

$$\lim_{k\to\infty} d(g(\omega, \eta_{n_k-1}(\omega)), g(\omega, \eta_{m_k+1}(\omega))) \le \varepsilon. \tag{35}$$

Since $m(k) \ge n(k) - 1$, so from (27), we have by (25),

$$\Phi_1(d(g(\omega, \xi_{m(k)+1}(\omega)), g(\omega, \xi_{n(k)}(\omega))))$$

$$= \Phi_1(d(F(\omega, (\xi_{m(k)}(\omega), \eta_{m(k)}(\omega))), F(\omega, (\xi_{m(k)-1}(\omega), \eta_{m(k)-1}(\omega)))))$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \xi_{m(k)}(\omega)), g(\omega, \xi_{n(k)-1}(\omega))), \\ d(g(\omega, \xi_{m(k)}(\omega)), g(\omega, \eta_{n(k)}(\omega))), \\ d(g(\omega, \xi_{m(k)}(\omega)), F(\omega, (\xi_{m(k)}(\omega), \eta_{m(k)}(\omega)))), \\ d(g(\omega, \xi_{n(k)-1}(\omega)), F(\omega, (\xi_{n(k)-1}(\omega), \eta_{n(k)-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \xi_{m(k)}(\omega)), F(\omega, (\xi_{n(k)-1}(\omega), \eta_{n(k)-1}(\omega)))) \\ \quad + d(g(\omega, \xi_{n(k)-1}(\omega)), F(\omega, (\xi_{m(k)}(\omega), \eta_{m(k)}(\omega))))] \end{pmatrix}$$

$$- \Psi_2\begin{pmatrix} d(g(\omega, \xi_{m(k)}(\omega)), g(\omega, \xi_{n(k)-1}(\omega))), \\ d(g(\omega, \eta_{m(k)}(\omega)), g(\omega, \eta_{n(k)}(\omega))), \\ d(g(\omega, \xi_{m(k)}(\omega)), F(\omega, (\xi_{m(k)}(\omega), \eta_{m(k)}(\omega)))), \\ d(g(\omega, \xi_{n(k)-1}(\omega)), F(\omega, (\xi_{n(k)-1}(\omega), \eta_{n(k)-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \xi_{m(k)}(\omega)), F(\omega, (\xi_{n(k)-1}(\omega), \eta_{n(k)-1}(\omega)))) \\ \quad + d(g(\omega, \xi_{n(k)-1}(\omega)), F(\omega, (\xi_{m(k)}(\omega), \eta_{m(k)}(\omega))))] \end{pmatrix}. \tag{36}$$

In addition, we have

$$\Phi_1(d(g(\omega, \eta_{n(k)}(\omega)), g(\omega, \eta_{m(k)+1}(\omega))))$$

$$= \Phi_1(d(F(\omega, (\eta_{m(k)-1}(\omega), \xi_{m(k)-1}(\omega))), F(\omega, (\eta_{m(k)}(\omega), \xi_{m(k)}(\omega)))))$$

$$\leq \Psi_1\begin{pmatrix} d(g(\omega, \eta_{n(k)-1}(\omega)), g(\omega, \eta_{m(k)}(\omega))), \\ d(g(\omega, \xi_{n(k)-1}(\omega)), g(\omega, \xi_{m(k)}(\omega))), \\ d(g(\omega, \eta_{m(k)}(\omega)), F(\omega, (\eta_{m(k)}(\omega), \xi_{m(k)}(\omega)))), \\ d(g(\omega, \eta_{n(k)-1}(\omega)), F(\omega, (\eta_{n(k)-1}(\omega), \xi_{n(k)-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \eta_{m(k)}(\omega)), F(\omega, (\eta_{n(k)-1}(\omega), \xi_{n(k)-1}(\omega)))) \\ \quad + d(g(\omega, \eta_{n(k)-1}(\omega)), F(\omega, (\eta_{m(k)}(\omega), \xi_{m(k)}(\omega))))] \end{pmatrix}$$

$$-\Psi_2\begin{pmatrix} d(g(\omega, \eta_{n(k)-1}(\omega)), g(\omega, \eta_{m(k)}(\omega))), \\ d(g(\omega, \xi_{n(k)-1}(\omega)), g(\omega, \xi_{m(k)}(\omega))), \\ d(g(\omega, \eta_{m(k)}(\omega)), F(\omega, (\eta_{m(k)}(\omega), \xi_{m(k)}(\omega)))), \\ d(g(\omega, \eta_{n(k)-1}(\omega)), F(\omega, (\eta_{n(k)-1}(\omega), \xi_{n(k)-1}(\omega)))), \\ \frac{1}{2}[d(g(\omega, \eta_{m(k)}(\omega)), F(\omega, (\eta_{n(k)-1}(\omega), \xi_{n(k)-1}(\omega)))) \\ \quad + d(g(\omega, \eta_{n(k)-1}(\omega)), F(\omega, (\eta_{m(k)}(\omega), \xi_{m(k)}(\omega))))] \end{pmatrix}. \tag{37}$$

Taking the limit as $k \to \infty$ using (26), (33), (34), (35) and the continuity of Ψ_1 and Ψ_2 in (36) and (37), we get the same equation (21), this holds if $\Psi_2(\varepsilon, \varepsilon, 0, 0, \varepsilon) = 0$, this implies that $\varepsilon = 0$, a contradiction since $\varepsilon > 0$. We deduce that $\{g(\omega, \xi_n(\omega))\}$ is a Cauchy sequence. Similarly, taking the limit as $k \to \infty$ in (37), we also get $\{g(\omega, \eta_{n+1}(\omega))\}$ is a Cauchy sequence. Since (X, d) is a complete metric space and $g(\omega \times X) = X$, so there exist $\zeta_0, \theta_0 \in \Theta$ such that

$$\lim_{n\to\infty} g(\omega, \xi_n(\omega)) = g(\omega, \zeta_0(\omega)) \text{ and } \lim_{n\to\infty} g(\omega, \eta_n(\omega)) = g(\omega, \theta_0(\omega)),$$

since $g(\omega, \zeta_0(\omega))$ and $g(\omega, \theta_0(\omega))$ are measurable, the functions $\zeta(\omega)$ and $\theta(\omega)$ defined by $\zeta(\omega) = g(\omega, \zeta_0(\omega))$ and $\theta(\omega) = g(\omega, \theta_0(\omega))$, are measurable. Thus,

$$\lim_{n\to\infty} g(\omega, \xi_n(\omega)) = \zeta(\omega) \text{ and } \lim_{n\to\infty} g(\omega, \eta_n(\omega)) = \theta(\omega). \tag{38}$$

From (38) and continuity of g,

$$\lim_{n\to\infty} g(\omega, g(\omega, \xi_n(\omega))) = g(\omega, \zeta(\omega)) \text{ and}$$
$$\lim_{n\to\infty} g(\omega, g(\omega, \eta_n(\omega))) = g(\omega, \theta(\omega)). \tag{39}$$

From (26) and commutativity of F and g,

$$\begin{aligned} g(\omega, g(\omega, \xi_{n+1}(\omega))) &= g(\omega, F(\omega, (\xi_n(\omega), \eta_n(\omega)))) \\ &= F(\omega, (g, (\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega)))) \end{aligned} \tag{40}$$

and

$$g(\omega, g(\omega, \eta_{n+1}(\omega))) = g(\omega, F(\omega, (\eta_n(\omega), \xi_n(\omega))))$$

$$= F(\omega, (g(\omega, \eta_n(\omega)), g(\omega, \xi_n(\omega)))). \tag{41}$$

We now show that if assumption (a) or (b) holds, then $g(\omega, \zeta(\omega)) = F(\omega, (\zeta(\omega), \theta(\omega)))$ and $g(\omega, \theta(\omega)) = F(\omega, (\theta(\omega), \zeta(\omega)))$.

Suppose at first that the assumption (a) holds. Then from (38), (40), (41) and continuity of F, we have

$$g(\omega, \zeta(\omega))$$

$$= \lim_{n\to\infty} g(\omega, g(\omega, \xi_{n+1}(\omega))) = \lim_{n\to\infty} F(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega))))$$

$$= F(\omega, (\lim_{n\to\infty} g(\omega, \xi_n(\omega)), \lim_{n\to\infty} g(\omega, \eta_n(\omega)))) = F(\zeta(\omega), \theta(\omega)),$$

$$g(\omega, \theta(\omega))$$

$$= \lim_{n\to\infty} g(\omega, g(\omega, \eta_{n+1}(\omega))) = \lim_{n\to\infty} F(\omega, g(\omega, \eta_n(\omega)), g(\omega, \xi_n(\omega)))$$

$$= F(\omega, (\lim_{n\to\infty} g(\omega, \eta_n(\omega)), \lim_{n\to\infty} g(\omega, \xi_n(\omega)))) = F(\theta(\omega), \zeta(\omega)).$$

From above equalities, we deduce that $(\zeta(\omega), \theta(\omega)) \in X \times X$ is a random coupled coincidence of F and g.

Suppose (b) holds. From (27), $g(\omega, \xi_n(\omega))$ is nondecreasing and $g(\omega, \xi_n(\omega)) \to g(\omega, \zeta(\omega))$ and $g(\omega, \eta_n(\omega))$ is nonincreasing and $g(\omega, \eta_n(\omega)) \to g(\omega, \theta(\omega))$, from (I) and (II), we have $g(\omega, \xi_n(\omega)) \leq g(\omega, \zeta(\omega))$ and $g(\omega, \eta_n(\omega)) \geq g(\omega, \theta(\omega))$ for all n. Hence, by using similar proof as in Theorem 3.1, we have

$$F(\omega, (\zeta(\omega), \theta(\omega))) = g(\omega, \zeta(\omega)).$$

Similarly, by the same manner, one can show that

$$F(\omega, (\theta(\omega), \zeta(\omega))) = g(\omega, \theta(\omega)).$$

Thus, $(\zeta(\omega), \theta(\omega)) \in X \times X$ is a random coupled coincidence of F and g.

From Theorem 3.2, we have the following coupled random fixed point which is a random analogue Theorem 11 of [30].

Theorem 3.3. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space. Suppose* $F : \Omega \times (X \times X) \to X$ *has the mixed monotone property and such that*

(i) $F(\omega, .)$ *is continuous for all* $\omega \in \Omega$,

(ii) $F(., v)$ *is measurable for all* $v \in X \times X$,

(iii)

$$\Phi_1(d(F(\omega, (x, y)), F(\omega, (u, v))))$$
$$\leq \Psi_1\begin{pmatrix} d(x, u), d(y, v), d(x, F(\omega, (x, y))), d(u, F(\omega, (u, v))), \\ \frac{1}{2}[d(x, F(\omega, (u, v))) + d(u, F(\omega, (x, y)))] \end{pmatrix}$$
$$- \Psi_2\begin{pmatrix} d(x, u), d(y, v), d(x, F(\omega, (x, y))), d(u, F(\omega, (u, v))), \\ \frac{1}{2}[d(x, F(\omega, (u, v))) + d(u, F(\omega, (x, y)))] \end{pmatrix}, \quad (42)$$

for $x, y, u, v \in X$, *for which* $x \geq u$ *and* $y \leq v$, *where* Ψ_1, Ψ_2 *are generalized altering distance functions and* $\Phi_1(x) = \Psi_1(x, x, x, x, x)$. *Also, suppose*

(a) *F is continuous or*

(b) *X has the following properties*:

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n*,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all n*.

If there exist measurable mappings $\eta_0, \xi_0 : \Omega \to X$ *such that*

$$\xi_0(\omega) \leq F(\omega, (\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$\eta_0(\omega) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega))) \text{ for all } \omega \in \Omega,$$

then there are measurable mappings $\zeta, \theta : \Omega \to X$ *such that*

$$F(\omega, (\zeta(\omega), \theta(\omega))) = \zeta(\omega) \text{ and } \theta(\omega) = F(\omega, (\theta(\omega), \zeta(\omega))) \text{ for all } \omega \in \Omega,$$

that is, *F has a random coupled fixed point*.

Proof. Taking $g : \Omega \times X \to X$ by $g(\omega, x) = x$ for all $\omega \in \Omega$ in Theorem 3.2, we obtain Theorem 3.3.

Now, a number of random coupled fixed point results may be obtained by assuming different forms for the functions Ψ_1 and Ψ_2. Here, we drive the following corollaries from Theorem 3.2 and Theorem 3.3.

Corollary 3.1. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space*. *Suppose* $F : \Omega \times (X \times X) \to X$ *and* $g : \Omega \times X \to X$ *are mappings such that*

(i) $F(\omega, .), g(\omega, .)$ *are continuous for all* $\omega \in \Omega$,

(ii) $F(., v), g(., x)$ *are measurable for all* $v \in X \times X$ *and* $x \in X$,

(iii) $F : \Omega \times (X \times X) \to X$ *and* $g : \Omega \times X \to X$ *are such that F has the mixed g-monotone property and*

$$d(F(\omega, (x, y)), F(\omega, (u, v)))$$

$$\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, u)) \\ + d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (u, v))) \\ + \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}, \tag{43}$$

for $x, y, u, v \in X$, $k \in [0, 1)$, *for which* $g(\omega, x) \geq g(\omega, u)$ *and* $g(\omega, y) \leq g(\omega, v)$. *Suppose* $F(X \times X) \subseteq g(X)$, $g(\omega, X) \to X$ *is continuous and commutes with F and suppose either*

(a) *F is continuous or*

(b) *X has the following properties*:

(I) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n*,

(II) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all n*.

If there exist measurable mappings $\eta_0, \xi_0 : \Omega \to X$ *such that*

$$g(\omega, \xi_0(\omega)) \leq F(\omega, (\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$g(\omega, \eta_0(\omega)) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega))) \text{ for all } \omega \in \Omega,$$

then there are measurable mappings $\varphi, \theta : \Omega \to X$ *such that*

$$g(\omega, \varphi(\omega)) = F(\omega, (\varphi(\omega), \theta(\omega))) \text{ and}$$

$$g(\omega, \theta(\omega)) = F(\omega, (\theta(\omega), \varphi(\omega))) \text{ for all } \omega \in \Omega,$$

that is, *F and g have a random coupled coincidence*.

Proof. Letting

$$\Psi_1(t_1, t_2, t_3, t_4, t_5) = \frac{1}{5}[t_1 + t_2 + t_3 + t_4 + t_5],$$

$$\Psi_2(t_1, t_2, t_3, t_4, t_5) = \frac{1-k}{5}[t_1 + t_2 + t_3 + t_4 + t_5]$$

and $\Phi_1(t) = t$ for all $t \in \Omega$ in Theorem 3.2, we obtain the proof.

Corollary 3.2. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space*. *Let* $F : \Omega \times (X \times X) \to X$ *be a mapping has the mixed monotone property and such that*

(i) $F(\omega, .)$ *is continuous for all* $\omega \in \Omega$,

(ii) $F(., v)$ *is measurable for all* $v \in X \times X$,

(iii) *assume there exists* $k \in [0, 1)$ *such that*

$$d(F(\omega, (x, y)), F(\omega, (u, v)))$$
$$\leq \frac{k}{5}\begin{pmatrix} d(x, u) + d(y, v) + d(x, F(\omega, (x, y))) + d(u, F(\omega, (u, v))) \\ + \frac{1}{2}[d(x, F(\omega, (u, v))) + d(u, F(\omega, (x, y)))] \end{pmatrix},$$

for $x, y, u, v \in X$, *for which* $x \geq u$ *and* $y \leq v$ *suppose either* F *is continuous or X has the following properties*:

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n*,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all n*.

If there exist measurable mappings $\eta_0, \xi_0 : \Omega \to X$ *such that*

$$\xi_0(\omega) \leq F(\omega(\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$\eta_0(\omega) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega))) \text{ for all } \omega \in \Omega,$$

then there are measurable mappings $\zeta, \theta : \Omega \to X$ *such that*

$$F(\omega(\zeta(\omega), \theta(\omega))) = \zeta(\omega) \text{ and } \theta(\omega) = F(\omega(\theta(\omega), \zeta(\omega))) \text{ for all } \omega \in \Omega,$$

that is, *F has a random coupled fixed point*, *that is*, $(\zeta(\omega), \theta(\omega)) \in X \times X$ *such that* $F(\omega, (\zeta(\omega), \theta(\omega))) = \zeta(\omega)$ *and* $\theta(\omega) = F(\omega, (\theta(\omega), \zeta(\omega)))$ *for all* $\omega \in \Omega$.

Proof. Letting $g = I$ (I is the identity mapping) for all $\omega \in \Omega$ in Corollary 3.1, we obtain the proof.

Corollary 3.3. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space*. *Let* $F : \Omega \times (X \times X) \to X$ *be a mapping has the mixed monotone property and such that*

(i) $F(\omega, .)$ *is continuous for all* $\omega \in \Omega$,

(ii) $F(., v)$ *is measurable for all* $v \in X \times X$,

(iii) *assume there exists* $k \in [0, 1)$ *such that*

$$d(F(\omega, (x, y)), F(\omega, (u, v)))$$

$$\leq k \max\begin{pmatrix} d(x, u), d(y, v), d(x, F(\omega, (x, y))), d(u, F(\omega, (u, v))), \\ \frac{1}{2}[d(x, F(\omega, (u, v))) + d(u, F(\omega, (x, y)))] \end{pmatrix},$$

for $x, y, u, v \in X$, *for which* $x \geq u$ *and* $y \leq v$ *suppose either F is continuous or X has the following properties*:

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n*,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all n*.

If there exist measurable mappings $\eta_0, \xi_0 : \Omega \to X$ *such that*

$$\xi_0(\omega) \leq F(\omega(\xi_0(\omega), \eta_0(\omega))) \text{ and}$$

$$\eta_0(\omega) \geq F(\omega, (\eta_0(\omega), \xi_0(\omega))) \text{ for all } \omega \in \Omega,$$

then there are measurable mappings $\zeta, \theta : \Omega \to X$ *such that*

$$F(\omega(\zeta(\omega), \theta(\omega))) = \zeta(\omega) \text{ and } \theta(\omega) = F(\omega, (\theta(\omega), \zeta(\omega))) \text{ for all } \omega \in \Omega,$$

that is, *F has a random coupled fixed point*, *that is*, $(\zeta(\omega), \theta(\omega)) \in X \times X$ *such that* $F(\omega, (\zeta(\omega), \theta(\omega))) = \zeta(\omega)$ *and* $\theta(\omega) = F(\omega, (\theta(\omega), \zeta(\omega)))$ *for all* $\omega \in \Omega$.

Proof. One can obtain the proof by taking $\Psi_1(t_1, t_2, t_3, t_4, t_5) = \max[t_1, t_2, t_3, t_4, t_5]$, $\Psi_2(t_1, t_2, t_3, t_4, t_5) = (1-k)\Psi_1(t_1, t_2, t_3, t_4, t_5)$ and $\Phi_1(t) = t$ for all $t \in \Omega$ in Theorem 3.3.

Example 3.1. Let $X = [0, +\infty)$ with the usual metric under $d(x, y) = |x - y|$ and ordering order $\leq$. Let $\Omega = [0, 1]$ and let Σ be the sigma measurable algebra subset of $[0, 1]$. Define $g : \Omega \times X \to X$ and $F : \Omega \times (X \times X) \to X$ as follows:

$$g(\omega, x) = (1-\omega^2)x \text{ and } F(\omega, x, y) = \begin{cases} \frac{1}{5}(1-\omega^2)(x-4y), & \text{if } x \geq 4y, \\ 0, & \text{if } x < y. \end{cases}$$

We will check that the contraction (43) of Corollary 3.1 is satisfied for all $x, y, u, v \in X$ satisfying $g(\omega, x) \geq g(\omega, u)$ and $g(\omega, y) \leq g(\omega, v)$ for all $\omega \in \Omega$. By taking $k = \frac{5}{6}$, we divide the proof into the following four cases:

(a) If $x \geq 4y$ and $u \geq 4v$, here we get $F(\omega, x, y) = \frac{(1-\omega^2)}{5}(x - 4y)$ and $F(\omega, u, v) = \frac{(1-\omega^2)}{5}(u - 4v)$,

$$
\begin{aligned}
& d(F(\omega, (x, y)), F(\omega, (u, v))) \\
&= (1-\omega^2)\left|\frac{x-4y}{5} - \frac{u-4v}{5}\right| \\
&= (1-\omega^2)\left(\frac{x-u}{5} + \frac{4(v-y)}{5}\right) \\
&= (1-\omega^2)\left(\frac{x-u}{6} + \frac{v-y}{6} + \frac{x-u}{30} + \frac{16(v-y)}{30} + \frac{3(v-y)}{30}\right) \\
&\leq (1-\omega^2)\left(\frac{x-u}{6} + \frac{v-y}{6} + \frac{x}{30} + \frac{4u}{30} + \frac{3v}{30}\right) \text{ since } v \leq \frac{1}{4}u \\
&\leq (1-\omega^2)\left(\frac{x-u}{6} + \frac{v-y}{6} + \frac{5x}{30} + \frac{4u}{30} + \frac{4v}{30}\right) \\
&= \frac{(1-\omega^2)}{6}\left(x - u + v - y + \frac{4x+4y}{5} + \frac{4u+4v}{30}\right) \text{ since } y \leq \frac{1}{4}x \\
&\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v)) \\ + d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (u, v))) \end{pmatrix} \\
&\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v)) \\ + d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (u, v))) \\ + \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}.
\end{aligned}
$$

(b) If $x \geq 4y$ and $u < 4v$, here we get $F(\omega, x, y) = \frac{(1-\omega^2)}{5}(x - 4y)$ and $F(\omega, u, v) = 0$,

$$d(F(\omega, (x, y)), F(\omega, (u, v)))$$

$$= (1-\omega^2)\frac{x-4y}{5} \leq (1-\omega^2)\frac{x}{5}$$

$$= (1-\omega^2)\left(\frac{x-u}{6} + \frac{u}{6} + \frac{x}{30}\right)$$

$$\leq (1-\omega^2)\left(\frac{x-u}{6} + \frac{u}{6} + \frac{5x}{30}\right)$$

$$\leq \frac{(1-\omega^2)}{6}\left(x - u + u + \frac{4x+4y}{5}\right) \text{ since } y \leq \frac{1}{4}x$$

$$\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, u), F(\omega, (u, v))) \\ + d(g(\omega, x), F(\omega, (x, y))) \end{pmatrix}$$

$$\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v)) \\ + d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (u, v))) \\ + \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}.$$

(c) If $x < 4y$ and $u \geq 4v$, here we get $F(\omega, x, y) = 0$ and $F(\omega, u, v) = \frac{(1-\omega^2)}{5}(u - 4v)$,

$$d(F(\omega, (x, y)), F(\omega, (u, v)))$$

$$= (1-\omega^2)\frac{u-4v}{5} \leq (1-\omega^2)\frac{u}{5} = (1-\omega^2)\left(\frac{u}{6} + \frac{u}{30}\right)$$

$$\leq (1-\omega^2)\left(\frac{x}{6} + \frac{u}{6} + \frac{u}{30}\right) \leq (1-\omega^2)\left(\frac{x}{6} + \frac{u}{6} + \frac{5u}{30}\right)$$

$$\leq \frac{(1-\omega^2)}{6}\left(x + u + \frac{4u + 4v}{5}\right) \text{ since } u \leq \frac{1}{4}v$$

$$\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (x, y))) \\ + d(g(\omega, u), F(\omega, (u, v))) \end{pmatrix}$$

$$\leq \frac{k}{5}\begin{pmatrix} d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v)) \\ + d(g(\omega, x), F(\omega, (x, y))) + d(g(\omega, u), F(\omega, (u, v))) \\ + \frac{1}{2}[d(g(\omega, x), F(\omega, (u, v))) + d(g(\omega, u), F(\omega, (x, y)))] \end{pmatrix}.$$

(d) If $x < 4y$ and $u < 4v$, here we get $F(\omega, (x, y)) = 0$ and $F(\omega, (u, v)) = 0$, it is trivial. Then from four cases, we have the hypotheses of Corollary 3.1 are verified. Thus, $(0, 0) \in X \times X$ is a random coupled coincidence and a random coupled fixed point of F and g.

As an application, it is easy to state a corollary of Theorem 3.3 involving a contraction of integral type.

Corollary 3.4. *Let F satisfy the conditions of Theorem* 3.3 *except that condition* (42) *be replaced by the following*: *there exists a positive Lebesgue integrable function* φ *on* R^+ *such that* $\int_0^{\varepsilon} \varphi(\omega, t)dt > 0$ *for each* $\varepsilon > 0$ *such that*

$$\int_0^{\Phi_1(d(F(\omega,(x,y)),F(\omega,(u,v))))} \varphi(\omega, t)dt$$

$$\leq \int_0^{\Psi_1\begin{pmatrix} d(x,u),\ d(y,v),\ d(x, F(\omega,(x,y))),\ d(u, F(\omega,(u,v))), \\ \frac{1}{2}[d(x, F(\omega,(u,v))) + d(u, F(\omega,(x,y)))] \end{pmatrix}} \varphi(\omega, t)dt$$

$$- \int_0^{\Psi_2\begin{pmatrix} d(x,u), d(y,v), d(x, F(\omega,(x,y))), d(u, F(\omega,(u,v))), \\ \frac{1}{2}[d(x, F(\omega,(u,v))) + d(u, F(\omega,(x,y)))] \end{pmatrix}} \varphi(\omega, t)dt.$$

Then F has a coupled random fixed point.

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References

[1] R. P. Agarwal, D. O'Regan and M. Sambandham, Random and deterministic fixed point theory for generalized contractive maps, Appl. Anal. 83 (2004), 711-725.

[2] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008), 109-116.

[3] H. Aydi, M. Postolache and W. Shatanawi, Coupled fixed point results for (ψ, φ)-weakly contractive mappings in ordered *G*-metric spaces, Comput. Math. Appl. 63 (2012), 298-309.

[4] I. Beg and N. Shahzad, Random fixed point theorems for nonexpansive and contractive-type random operators on Banach spaces, J. Appl. Math. Stoch. Anal. 7 (1994), 569-580.

[5] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. Theor. 65 (2006), 1379-1393.

[6] B. S. Choudhury, Convergence of a random iteration scheme to a random fixed point, J. Appl. Math. Stoch. Anal. 8 (1995), 139-142.

[7] B. S. Choudhury and M. Ray, Convergence of an iteration leading to a solution of a random operator equation, J. Appl. Math. Stoch. Anal. 12 (1999), 161-168.

[8] B. S. Choudhury and P. N. Dutta, A unified fixed point result in metric spaces involving a two variable function, Filomat 14 (2000), 43-48.

[9] B. S. Choudhury, A common unique fixed point result in metric spaces involving generalized altering distances, Math. Commun. 10 (2005), 105-110.

[10] B. S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling 54 (2011), 73-79.

[11] B. S. Choudhury, N. Metiya and M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. 2013 (2013), 152.

[12] Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet and W. Shatanawi, Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, Fixed Point Theory Appl. 2012 (2012), 8.

[13] Lj. B. Ćirić, S. N. Ješic and J. S. Ume, On random coincidence for a pair of measurable mappings, J. Inequal. Appl. 2006 (2006), 1-12.

[14] Lj. B. Ćirić, M. M. Milovanović-Arandjelović and N. T. Nikolić, On random coincidence and fixed points for a pair of multi-valued and single valued mappings, Ital. J. Pure Appl. Math. 23 (2008), 37-44.

[15] Lj. B. Ćirić, N. Cakić, M. Rajović and J. S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2008 (2008), 1-11.

[16] O. Hadzić, A random fixed point theorem for multi-valued mappings of Ćirić's type, Mat. Vesnik 3 (1979), 16-31.

[17] O. Hanś, Reduzierende Zufällige transformationen, Czechoslovak Math. J. 7 (1957), 154-158.

[18] O. Hanś, Random operator equations, Proc. 4th Berkeley Symp. Mathematical Statistics and Probability, Vol. II, Part I, University of California Press, Berkeley, 1961, pp. 185-202.

[19] C. J. Himmelberg, Measurable relations, Fund. Math. 87 (1975), 53-72.

[20] N. J. Huang, A principle of randomization of coincidence points with applications, Appl. Math. Lett. 12 (1999), 107-113.

[21] N. Hussain, Common fixed points in best approximation for Banach operator pairs with Ćirić type I-contractions, J. Math. Anal. Appl. 338 (2008), 1351-1363.

[22] S. Itoh, Random fixed-point theorems with an application to random differential equations in Banach spaces, J. Math. Anal. Appl. 67 (1979), 261-273.

[23] E. Karapinar, Couple fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59 (2010), 3656-3668.

[24] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984), 1-9.

[25] V. Lakshmikantham and Lj. B. Ćirić, Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, Stoch. Anal. Appl. 27 (2009), 1246-1259.

[26] V. Lakshmikantham and Lj. B. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[27] T. C. Lin, Random approximations and random fixed point theorems for non-self maps, Proc. Amer. Math. Soc. 103 (1988), 1129-1135.

[28] S. G. Matthews, Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., Vol. 728, New York Academy of Sciences, New York, 1994, pp. 183-197.

[29] E. J. McShane and R. B. Warfield, Jr., On Filippov's implicit functions lemma, Proc. Amer. Math. Soc. 18 (1967), 41-47.

[30] H. K. Nashine and H. Aydi, Coupled fixed point theorems for contractions involving altering distances in ordered metric spaces, Math. Sci. 20 (2013), 1-8.

[31] J. J. Nieto and R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223-239.

[32] J. J. Nieto, R. L. Pouso and R. Rodriguez-Lopez, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc. 135 (2007), 2505-2517.

[33] N. S. Papageorgiou, Random fixed point theorems for multifunctions, Math. Japonica 29 (1984), 93-106.

[34] S. Radenović, Remarks on some recent coupled coincidence point results in symmetric *G*-metric spaces, J. Operators 2013 (2013), 8 pp, Article ID 290525.

[35] S. Radenović, Remarks on some coupled coincidence point results in partially ordered metric spaces, Arab J. Math. Sci. 20 (2014), 29-39.

[36] K. P. R. Rao, G. R. Babu and D. V. Babu, Common fixed point theorems through generalized altering distance functions, Math. Commun. 13 (2008), 67-73.

[37] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[38] R. T. Rockafellar, Measurable dependence of convex sets and functions in parameters, J. Math. Anal. Appl. 28 (1969), 4-25.

[39] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508-4517.

[40] S. Sedghi, I. Altun and N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Anal. 72 (2010), 1298-1304.

[41] V. M. Sehgal and S. P. Singh, On random approximations and a random fixed point theorem for set valued mappings, Proc. Amer. Math. Soc. 95 (1985), 91-94.

[42] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Math. Comput. Modelling 55 (2012), 680-687.

[43] W. Shatanawi, On *w*-compatible mappings and common coupled coincidence point in cone metric spaces, Appl. Math. Lett. 25 (2012), 925-931.

[44] A. Špaček, Zufällige Gleichungen, Czechoslovak Math. J. 5 (1955), 462-466.

[45] D. H. Wagner, Survey of measurable selection theorems, SIAM J. Control Optim. 15 (1977), 859-903.